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MULTIDIMENSIONAL IFRA PROCESSES.(U)

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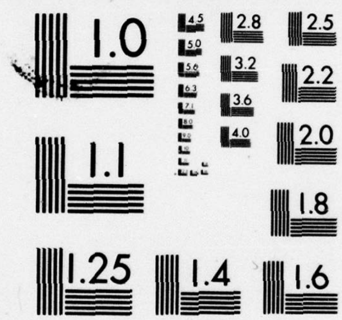

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Multidimensional IFRA Processes

by

Henry W. Block and Thomas H. Savits

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Multidimensional IFRA Processes[1]

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Abstract

Two types of multidimensional processes are defined. The first of these generalizes a univariate IFRA process due to Ross and relates to a multivariate concept of IFRA due to Esary and Marshall. The second of these relates to a multivariate concept of IFRA due to the present authors.

Decompositions for multistate monotone structure functions are given and behavior of nonincreasing stochastic processes such as those given above is analyzed. Various coherence assumptions for multistate systems are also analyzed.

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§0. Introduction

Ross [7] has defined a univariate nondecreasing process to be IFRA (increasing failure rate average) if certain lifetimes associated with the process are IFRA. See Barlow and Proschan [2] for a discussion of IFRA lifetimes. Extensions of IFRA to multivariate lifetimes have been proposed by Block and Savits [3] and Esary and Marshall [4]. In this paper the univariate concept of Ross is extended to multidimensional processes and related to IFRA multivariate lifetimes.

In Section 1 a characterization of univariate IFRA processes is given. The Ross concept of IFRA processes is extended to vector processes and an alternate form is derived. A closure theorem and various properties are established for these processes. It is shown in Theorem 2.4 of Section 2, that lifetimes associated with these processes satisfy the condition that any monotone system formed with these lifetimes is IFRA in the univariate sense. Furthermore this property characterizes such processes. This property, called Condition B in Esary and Marshall [4], was one of the definitions of multivariate IFRA discussed by those authors. Another type of multidimensional IFRA process is defined. For this process, the associated lifetime satisfy the MIFRA property of Block and Savits [3].

In Section 3, decompositions of multistate structure functions are given. The main result, Theorem 3.10, is that a multistate structure function for a system whose components can operate at a finite number of levels can be written as a sum of certain binary structure functions. Using these ideas, the behavior of nonincreasing stochastic processes (such as those discussed in Sections 1 and 2) is analyzed. Various coherence assumptions for multistate systems proposed by El-Neweihi, Proschan and Sethuraman [5] and Griffith [6] are analyzed in Section 4.

§1. IFRA processes and the IFRA closure theorem.

Let $X(t)$ be a nonnegative, nonincreasing right-continuous random process. According to Ross [7], the process $X(t)$ is called an IFRA process if and only if the random variable

(1.1) $$T_a = \inf \{t > 0 : X(t) \leq a\}$$

is IFRA for every $a \geq 0$. Equivalently, we have the alternative characterization below.

(1.2) Theorem. $X(t)$ is an IFRA process if and only if

(1.3) $$E\,[h(X(t))] \leq E^{1/\alpha}\,[h^{\alpha}(X(\alpha t))]$$

for all nonnegative nondecreasing functions h and all $0 < \alpha \leq 1$, $t \geq 0$.

Proof. First assume that $X(t)$ is an IFRA process and consider h of the form $h(x) = I_{(a,\infty)}(x)$, $a \geq 0$. Since, by right-continuity, $X(t) > a$ if and only if $T_a > t$, we have

$$E\,[h(X(t))] = P(X(t) > a) = P(T_a > t)$$

$$\leq P^{1/\alpha}(T_a > \alpha t) = P^{1/\alpha}(X(\alpha t) > a) = E^{1/\alpha}\,[h^{\alpha}(X(\alpha t))]$$

for all $0 < \alpha \leq 1$, $t \geq 0$. Now consider h of the form $h(x) = I_{(a,\infty)}(x)$, $a > 0$ (the case $a = 0$ is clear). Since $I_{(a-1/n,\infty)}(x) \downarrow h(x)$, the inequality (1.3) is also valid for such h. The general result now follows by taking nonnegative linear combinations of such functions and passing to the limit as in Block and Savits [3].

Conversely, if (1.3) is true, then (1.1) follows by taking $h(x) = I_{(a,\infty)}(x)$.

Ross [7] proved the IFRA closure theorem under the assumption of independent components. We obtain the same results without the assumption of independence. First, however, we need some definitions.

(1.4) Definition. An upper set $U \subset \mathbb{R}$ is a subset having the property that if $\underline{x} \in U$ and $\underline{y} \geq \underline{x}$, then $\underline{y} \in U$. If in addition U is an open subset, we call U an upper domain.

Now let $\underline{X}(t) = (X_1(t), \ldots, X_n(t))$ be a vector-valued stochastic process. We assume that $\underline{X}(t)$ is nonnegative, nonincreasing and right-continuous.

(1.5) Definition. $\underline{X}(t)$ is said to be a (vector-valued) IFRA process if and only if for every upper domain U, the random variable

$$T_U = \inf \{t \geq 0 : \underline{X}(t) \notin U\}$$

is IFRA.

Clearly this includes the IFRA class considered by Ross [7] in the case $n = 1$. Again, as in (1.2), we have the alternative characterization given below.

(1.6) Theorem. $\underline{X}(t)$ is a (vector-valued) IFRA process if and only if

$$E\,[h(\underline{X}(t))] \leq E^{1/\alpha}\,[h^{\alpha}(\underline{X}(\alpha t))]. \tag{1.7}$$

For all continuous nonnegative nondecreasing functions h and all $0 < \alpha \leq 1$, $t \geq 0$.

Proof. The proof is very similar to (1.2): first show that (1.7) is true if $h(\underline{x}) = I_U(\underline{x})$ for U an upper domain and then use the argument in Block and Savits [3] for general h.

(1.8) Remark. The restriction to continuous nonnegative nondecreasing n in Theorem 1.6 is just a technical convenience. As in Block and Savits [3], we can show that if (1.7) is valid for all continuous h, then it is valid for all Borel measurable nonnegative nondecreasing h.

The next theorem describes some properties of the class of IFRA processes. We henceforth dispense with the adjective vector-valued.

(1.9) Theorem

(i) If $\underline{X}(t)$ is an IFRA process and $\psi_1, \ldots, \psi_k$ are continuous nonnegative nondecreasing functions, then $(\psi_1(\underline{X}(t)), \ldots, \psi_k(\underline{X}(t)))$ is an IFRA process.

(ii) If $(X_1(t), \ldots, X_n(t))$ and $(Y_1(t), \ldots, Y_m(t))$ are IFRA processes which are independent at each time t, then $(X_1(t), \ldots, X_n(t), Y_1(t), \ldots, Y_m(t))$ is an IFRA process.

(iii) If $\underline{X}_n(t)$, $n = 1,2,\ldots$, are IFRA processes and $\underline{X}_n(t) \to \underline{X}(t)$ weakly for each t, then $\underline{X}(t)$ is an IFRA process provided it is also nonnegative, nondecreasing and right-continuous.

Proof. The proofs are clear and are left to the reader.

Let Φ be a multistate monotone structure function of n-components, i.e., $\Phi(\underline{x}) = \Phi(x_1, \ldots, x_n)$ is continuous nonnegative and nondecreasing in each argument.

(1.10) Corollary

(i) (IFRA closure theorem). If Φ is a multistate monotone structure function and $\underline{X}(t)$ is an IFRA process, then $\Phi(\underline{X}(t))$ is an IFRA process.

(ii) (Convolution theorem). If $(X_1(t), \ldots, X_n(t))$ and $(Y_1(t), \ldots, Y_n(t))$ are IFRA processes which are independent at each time t, then $(X_1(t) + Y_1(t), \ldots, X_n(t) + Y_n(t))$ is an IFRA process.

(iii) If $(X_1(t),\ldots, X_n(t))$ is an IFRA process and $J \subset \{1,\ldots, n\}$, then $(X_j(t) : j \in J)$ is an IFRA process.

§2. IFRA processes and multivariate IFRA concepts.

Let T be a nonnegative random variable and X(t) its indicator process, i.e., $X(t) = I_{(t,\infty)}(T)$. Then clearly X(t) is an IFRA process if and only if T is an IFRA random variable since

$$T_a = \inf\{t \geq 0 : X(t) \leq a\} = \begin{cases} 0 & \text{if } a \geq 1 \\ T & \text{if } 0 \leq a < 1. \\ +\infty & \text{if } a < 0 \end{cases}$$

Now let $(T_1,\ldots, T_n)$ be a nonnegative random vector. If we assume that $(T_1,\ldots, T_n)$ is MIFRA in the sense of Block and Savits [3], then there are many ways of constructing IFRA processes. For example, suppose that $\phi(t; x_1,\ldots, x_n)$, $t, x_1,\ldots, x_n \geq 0$, is nonnegative, Borel measureable and nondecreasing in $\underline{x}$ for fixed t, right-continuous and nonincreasing in t for fixed $\underline{x}$, and satisfies

$$\phi(t; x_1/\alpha,\ldots, x_n/\alpha) \leq \phi(\alpha t; x_1,\ldots, x_n)$$

for all $0 < \alpha \leq 1$, $t \geq 0$, $\underline{x} \in \mathbb{R}^n_+$. Then $X(t) = \phi(t; T_1,\ldots, T_n)$ is an IFRA process. Indeed, let h be any continuous nonnegative nondecreasing function. Then

$$E[h(X(t))] = E[h(\phi(t; T_1,\ldots, T_n))] \leq E^{1/\alpha}[h^\alpha(\phi(t; T_1/\alpha,\ldots, T_n/\alpha))]$$

$$E^{1/\alpha}[h^\alpha(\phi(\alpha t; T_1,\ldots, T_n))] = E^{1/\alpha}[h^\alpha(X(\alpha t))].$$

In particular, if $x_{1:n} \le x_{2:n} \le \cdots \le x_{n:n}$ is a reordering of $x_1, \ldots, x_n$, then

$$\phi(t; x_1, \ldots, x_n) = \begin{cases} n & \text{if} \quad 0 \le t < x_{1:n} \\ n-k & \text{if} \quad x_{k:n} \le t < x_{k+1:n},\ k = 1, \ldots, n-1 \\ 0 & \text{if} \quad t \ge x_{n:n} \end{cases}$$

has the desired properties.

(2.1) Example. Let (S,T) be MIFRA and set

$$X(t) = \begin{cases} 2 & \text{if} \quad 0 \le t < \min (S,T) \\ 1 & \text{if} \quad \min (S,T) \le t < \max (S,T) \\ 0 & \text{if} \quad t \ge \max (S,T) \end{cases}$$

Then X(t) is an IFRA process.

(2.2) Example. Let (S,T) have the distribution with joint density

$$f(s,t) = \begin{cases} 30 & \text{if} \quad 3/8 < s < 1/2, \quad 3/4 < t < 1 \\ 1 & \text{if} \quad 1/8 < s < 3/8, \quad 1/2 < t < 2/3 \\ 2 & \text{if} \quad 0 < s < 1/8, \quad 2/3 < t < 3/4\,. \end{cases}$$

Then S and T are IFRA and $S \le T$ with probability one. Consequently,

$$X(t) = \begin{cases} 2 & \text{if} \quad 0 \le t < S \\ 1 & \text{if} \quad S \le t < T \\ 0 & \text{if} \quad T \le t \end{cases}$$

is an IFRA process. But (S,T) is not MIFRA since $P(S > t,\ T > 4t)$ has support that is not an interval and we know that if (S,T) was MIFRA, then min (S, T/4) would be IFRA.

Recall that from Esary and Marshall [4], a nonnegative random vector $(T_1,\ldots, T_n)$ satisfies condition (B) if and only if $\tau(T_1,\ldots, T_n)$ is IFRA for every life function τ corresponding to a monotone binary structure function ϕ. This condition can be characterized in terms of IFRA processes as follows.

(2.3) Theorem. Let $\underline{T} = (T_1,\ldots, T_n)$ be a nonnegative random vector. Then $\underline{T}$ satisfies condition (B) if and only if the indicator process $\underline{X}(t) = (X_1(t),\ldots, X_n(t))$, where $X_i(t) = I_{(t,\infty)}(T_i)$, is an IFRA process.

Proof. Suppose that the indicator process $\underline{X}(t)$ is an IFRA process. If ϕ is a binary monotone structure function with corresponding life function τ, then

$$P(\tau > t) = E[\phi(X_1(t),\ldots, X_n(t))]$$

$$\leq E^{1/\alpha}[\phi^{\alpha}(\underline{X}(\alpha t))] = P^{1/\alpha}(\tau > \alpha t)$$

and so τ is IFRA.

Now suppose that $\underline{T} = (T_1,\ldots, T_n)$ satisfies condition (B) and let U be any upper domain in $\mathbb{R}^n$. If $\underline{x} = (x_1,\ldots, x_n)$ is any binary vector of ones and zeros, set

$$\phi_U(\underline{x}) = \begin{cases} 1 & \text{if } \underline{x} \in U \\ 0 & \text{otherwise} \end{cases}.$$

Then ϕ_U is a binary monotone structure function. Furthermore, if τ_U is its corresponding life function and $T_U = \inf\{t : \underline{X}(t) \notin U\}$, then $T_U = \tau_U$. But by assumption, τ_U is IFRA and so T_U is also IFRA. Consequently, $\underline{X}(t)$ is an IFRA process.

Theorem (2.3) extends to the general case as follows. Let $\underline{X}(t)$ be an IFRA process and let U be an upper domain in $\mathbb{R}_n$. Then, according to Block and Savits [3], there exist fundamental upper domains U_ℓ such that $\bigcup_{\ell=1}^{\infty} U_\ell = U$

and $T_{U_\ell} \uparrow T_U$. Since the IFRA class is closed under weak limits, it suffices to show that T_U is IFRA for every fundamental upper domain U. But by definition, U is a fundamental upper domain if and only if $U = \bigcup_{\ell=1}^{p} U_\ell$, where $U_\ell = \{\underline{x} \in \mathbb{R}^n : x_i > z_{i\ell}\}$ and $z_{i\ell}$ are real numbers for $1 \leq i \leq n$, $1 \leq \ell \leq p$. Clearly $T_U = \max_{1\leq \ell \leq p} T_{U_\ell}$, and if we set $T_{iz} = \inf\{t \geq 0 : X_i(t) \leq z_{i\ell}\}$, then

$$T_U = \max_{1\leq\ell\leq p} \min_{1\leq i\leq n} T_{iz_{i\ell}} .$$

Consequently we may state the following result.

(2.4) <u>Theorem</u>. $\underline{X}(t)$ is an IFRA process if and only if every finite collection of $\{T_{iz} : 1 \leq i \leq n, z \in \mathbb{R}\}$ satisfies condition (B) of Esary and Marshall.

(2.5) <u>Corollary</u>. In the finite state case, i.e., $X_i(t) \in \{0,1,\ldots, M\}$ for all $t \geq 0$, $i = 1,\ldots, n$, $\underline{X}(t)$ is an IFRA process if and only if $\{T_{ij} : 1 \leq i \leq n, 0 \leq j < M\}$ satisfies condition (B), where

$$T_{ij} = \inf\{t \geq 0 : X_i(t) \leq j\}$$

Clearly, in the finite state case, if the finite collection $\{T_{ij} : 1 \leq i \leq n, 0 \leq j < M\}$ are MIFRA, then they satisfy condition (B). This leads to the following definition.

(2.6) <u>Definition</u>. Let $\underline{X}(t)$ be a nonnegative nondecreasing right-continuous process. Then we say that $\underline{X}(t)$ is an <u>MIFRA process</u> if and only if for every finite collection $U_1,\ldots, U_m$ of upper domains in $\mathbb{R}^n$, the random vector $(T_{U_1},\ldots, T_{U_m})$ is MIFRA.

As example (2.2) shows, there exist IFRA processes which are not MIFRA. The analogous result to Theorem (2.4) is stated below.

(2.7) Theorem. $\underline{X}(t)$ is an MIFRA process if and only if every finite collection of $\{T_{iz} : 1 \le i \le n, z \in \mathbb{R}\}$ is MIFRA.

(2.8) Remark. Note that for IFRA processes, the upper domains are defined with respect to the state space, whereas for MIFRA vectors, the upper domains are defined with respect to the time space.

§3. Decompositions of multistate structure functions.

Multistate structure functions have been studied by Barlow [1], El-Newechi, Proschan and Sethuraman (henceforth EPS) [5] and Griffith [6]. EPS and Griffith make a variety of coherence assumptions which we shall not make here. Barlow uses a decomposition of the binary structure function to define a multistate structure function. In this section we obtain a general decomposition of this type for finite multistate structure functions along with various properties of these functions. The first two results below were proven by Griffith [6]. The notation of Barlow and Proschan [2] and EPS [5] is used here.

Let $S = \{0,1,\ldots, M\}$ and $\phi : S^n \to S$ be a nondecreasing function. The values taken by ϕ will represent the system performance and for each i, x_i will denote the performance of the ith component. We distinguish M+1 performance levels ranging from perfect functioning (level M) to complete failure (level 0). The assumption that ϕ is nondecreasing corresponds to the notion that the system does not perform worse if its components are improved.

We first consider a function $\phi : S^n \to S$ where $S = \{0,1,\ldots, M\}$ and give conditions for ϕ to be nondecreasing.

(3.1) Theorem. ϕ is nondecreasing if and only if

$$\phi(\underline{x} \vee \underline{y}) \ge \phi(\underline{x}) \vee \phi(\underline{y})$$

if and only if

$$\Phi(\underline{x} \wedge \underline{y}) \le \Phi(\underline{x}) \wedge \Phi(\underline{y}).$$

(3.2) **Theorem.** Let Φ be nondecreasing. Then for all $\underline{x} = (x_1, \dots, x_n) \in S^n$

(i) $\min_i x_i \le \Phi(\underline{x})$ if and only if $\Phi(\underline{k}) \ge k$ all $k \in S$.

(ii) $\Phi(\underline{x}) \le \max_i x_i$ if and only if $\Phi(\underline{k}) \le k$ all $k \in S$.

Consequently, $\min_i x_i \le \Phi(x) \le \max_i x_i$ if and only if $\Phi(\underline{k}) = k$ for all $k \in S$.

An easy consequence of the monotonicity assumption is stated below

(3.3) **Theorem.** Let Φ be nondecreasing. Then

(i) $\max_i \Phi((x_i)_i; \underline{0}) \le \Phi(\underline{x}) \le \min_i \Phi((x_i)_i, \underline{M})$

(ii) $\Phi(\min_i x_i) \le \Phi(\underline{x}) \le \Phi(\max_i x_i)$.

Furthermore these bounds are not compatible in the sense that there exist systems Φ for which (i) is a better bound than (ii) and vice-versa.

For the next results besides assuming Φ is nondecreasing, we impose the condition that $\Phi(\underline{0}) = \underline{0}$ and $\Phi(\underline{M}) = \underline{M}$. This merely states that if all components fail, the system fails and if all components are functioning perfectly, the system functions perfectly. We do not make the assumption imposed by EPS and Griffith that $\Phi(\underline{k}) = k$ for $k = 1, \dots, M-1$. We will call such a function Φ a multistate monotone structure function (MMS).

(3.4) **Definition.** A vector $\underline{x}$ is called an upper (lower) vector for level k of a MMS if $\Phi(\underline{x}) \ge k$ ($\Phi(\underline{x}) \le k$). It is called a critical upper (lower) vector for level k if in addition $\underline{y} < \underline{x}$ implies $\Phi(\underline{y}) < k$ (if $\underline{y} > \underline{x}$ implies $\Phi(\underline{y}) > k$).

The set of all critical upper (lower) vectors for level k is denoted by U_k or $U_k(\Phi)$ if necessary (L_k or $L_k(\Phi)$). If $\underline{x} \in U_k$, $k = 1,2,\ldots, M$, let

$$U_k(\underline{x}) = U_k(\Phi;\underline{x}) = \{(i,x_i) : x_i \neq 0\}.$$

if $\underline{x} \in L_k$, $k = 0,1,\ldots, M-1$, let

$$L_k(\underline{x}) = L_k(\Phi;\underline{x}) = \{(i,x_i) : x_i \neq M\}$$

as we will see, these sets play the role of min path sets and min cut sets respectively.

As usual, the concept of duality changes upper vector concepts to lower vector concepts. More precisely, if Φ is an MMS, then $\Phi^D(\underline{x}) = M-\Phi(\underline{M}-\underline{x})$ is also an MMS called the dual of Φ. The proofs of the following two results are obvious.

(3.5) Theorem. The vector $\underline{x}$ is an upper vector for level k of Φ if and only if $\underline{M}-\underline{x}$ is a lower vector for level M-k of Φ^D. Furthermore, $\underline{x} \in U_k(\Phi)$ if and only if $\underline{M}-\underline{x} \in L_{M-k}(\Phi^D)$.

(3.6) Theorem. For $k > 0$ $\Phi(\underline{x}) \geq k$ if and only if $\underline{x} \geq \underline{x}^0$ for some $\underline{x}^0 \in U_k$.

(3.7) Remark. The assumption $\Phi(\underline{M}) = M$ implies $U_k \neq \phi$ for $k = 1,\ldots, M$ and $U_M \neq \phi$ implies $\Phi(\underline{M}) = M$.

Now we define the binary function ϕ_k for M·n binary variables $\underline{y} = (y_{ij} : 1 \leq i \leq n, 1 \leq j \leq M)$ by

$$(3.8) \qquad \phi_k(\underline{y}) = \max_{\underline{x} \in U_k} \ \min_{(i,j) \in U_k(\underline{x})} y_{ij}, \quad k = 1,\ldots, M.$$

Although this function is defined for all M·n values of $\underline{y}$, we are only interested in this function on the domain given by the image of the following function.

We define $\alpha : S^n \to \{0,1\}^{M \cdot n}$ by $\alpha(\underline{x}) = (\alpha_{ij}(\underline{x}) : 1 \le i \le n,\ 1 \le j \le M)$, where $\underline{x} \in S^n$ and $\alpha_{ij}(\underline{x}) = 1$ if $x_i \ge j$ and 0 otherwise.

(3.9) Lemma. For $k > 0$ $\phi(\underline{x}) \ge k$ if and only if $\phi_k(\alpha(\underline{x})) = 1$.

(3.10) Theorem. $\phi(\underline{x}) = \sum_{k=1}^{M} \phi_k(\alpha(\underline{x}))$.

Since the proofs are straightforward, we omit them. Theorem (3.10) is a type of decomposition result analogous to those using min path and min cut sets in the binary case.

(3.11) Remarks. (i) Note that $\phi_1(\alpha(\underline{x})) \ge \phi_2(\alpha(\underline{x})) \ge \cdots \ge \phi_M(\alpha(\underline{x}))$; equivalently, $\phi_1 \ge \phi_2 \ge \cdots \ge \phi_M$ on $\Delta = \alpha(S^n) = \{\underline{y} = (y_{ij}) : \text{if } y_{ij} = 1, \text{ then } y_{i\ell} = 1 \text{ for all } \ell = 1,\ldots, j-1\}$.

(ii) For a binary monotone structure function $\phi(\underline{x})$ of n binary variables, if $\underline{x}$ is a min path vector (see Barlow and Proschan [2]) then the min path set corresponding to $\underline{x}$ is defined by $C_1(\underline{x}) = \{i : x_i = 1\}$. Designating the min path sets of ϕ by $P_1, P_2, \ldots, P_p$, the min path decomposition is given by

$$\phi(\underline{x}) = \max_{1 \le j \le p} \min_{i \in P_j} x_i .$$

The sets $\{U_k(\underline{x}),\ \underline{x} \in U_k\}$ play a similar role here. Furthermore, in terms of the binary monotone structure function of binary variables, $\phi_k(\underline{y})$, where $\underline{y}$ is restricted to Δ, if $\underline{y}$ is a min path vector, then we could associate the set $C_k(\underline{y}) = \{(i,j) : y_{ij} = 1,\ y_{i,j+1} = 0\} = U_k(\underline{x})$ where $\alpha(\underline{x}) = \underline{y}$. Thus in this sense the $U_k(\underline{x})$ are also the min path sets of $\phi_k(\underline{y})$. We shall call the sets $\{U_k(x),\ x \in U_k\}$ the min path sets of ϕ_k. If P is a min path set of ϕ_k and $(i,j) \in P$, then $(i,\ell) \notin P$ for any $\ell \ne j$.

(iii) If $\phi_1, \ldots, \phi_M$ are binary monotone structure functions of the binary variables $(y_{ij} : 1 \le i \le n, 1 \le j \le M)$ restricted to Δ and they satisfy (i) above, then $\phi(\underline{x}) = \sum_{k=1}^{M} \phi_k\,(\alpha(\underline{x}))$ is a multistate monotone structure function whose decomposition yields precisely the $\phi_1, \ldots, \phi_M$.

(3.12) <u>Example</u>. An example will serve to illustrate the procedure. Let $\phi : \{0,1,2\}^2 \to \{0,1,2\}$ with $0 = \phi(0,0) = \phi(1,0)$, $1 = \phi(0,1) = \phi(0,2) = \phi(1,2) = \phi(1,1)$, $2 = \phi(2,0) = \phi(2,1) = \phi(2,2)$. Then

$$U_1 = \{(2,0),\ (0,1)\}, \quad U_2 = \{(2,0)\}$$

and

$$U_1(2,0) = \{(1,2)\} = U_2(2,0),$$

$$U_1(0,1) = (2,1) .$$

Then

$$\phi_2(y) = \max_{\underline{x} \in U_2} \ \min_{(i,j) \in U_2(x)} y_{ij} = \min_{(i,j) \in U_2(2,0)} y_{ij} = y_{12}$$

and

$$\phi_1(\underline{y}) = \max_{\underline{x} \in U_1} \ \min_{(i,j) \in U_1(\underline{x})} y_{ij} = \max(y_{21}, y_{12})$$

where we have

$\underline{x}$	$\alpha(\underline{x})$	$\phi_1(\alpha(\underline{x}))$	$\phi_2(\alpha(\underline{x}))$	$\phi(x) = \sum_{k=1}^{2} \phi_k(\alpha(\underline{x}))$
(0,0)	(0,0,0,0)	0	0	0
(0,1)	(0,0,1,0)	1	0	1
(0,2)	(0,0,1,1)	1	0	1
(1,0)	(1,0,0,0)	0	0	0
(1,1)	(1,0,1,0)	1	0	1
(1,2)	(1,1,0,0)	1	1	2
(2,0)	(1,1,0,0)	1	1	2
(2,2)	(1,1,1,1)	1	1	2

Notice, for example, that to the critical upper vector of level 1, (2,0) corresponds the set $U_1(2,0) = \{(1,2)\}$. Also notice that $\alpha(2,0) = (1,1,0,0)$ is a min path vector for ϕ_1 in the sense that $\phi_1(1,1,0,0) = 1$ but $\phi_1(1,0,0,0) = 0$ ((0,1,0,0) $\notin \Delta$ so we don't consider it). The corresponding min path set is $C_1(1,1,0,0) = \{(1,2)\}$ which is $U_1(2,0)$.

A similar decomposition can be obtained using critical lower vectors. More precisely, define the binary structure function ψ_k of the M·n binary variable $\underline{z} = (z_{ij} : 1 \leq i \leq n, 0 \leq j \leq M\text{-}1)$ by

$$\psi_k(\underline{z}) = \min_{\underline{x} \in L_k} \max_{(i,j) \in L_k(\underline{x})} z_{ij}$$

for k = 0,1,..., M-1. As in the previous case we restrict the domain of $\psi_k(\underline{z})$ to the image of $\beta : S^n \to \{0,1\}^{M,n}$ where $\beta(\underline{x}) = (\beta_{ij}(\underline{x}) : 1 \leq i \leq n, 0 \leq j \leq M\text{-}1)$ and $\beta_{ij}(\underline{x}) = 0$ if $x_i \leq j$ and 1 otherwise.

(3.13) Theorem. $\phi(\underline{x}) = \sum_{k=0}^{M-1} \psi_k \;(\beta(\underline{x}))$

Proof. The proof is most easily obtained by duality arguments.

We now consider the stochastic behavior. Let $X_i(t)$ be a right-continuous nonincreasing stochastic process with values in S; i.e., $X_i(t)$ represents the statistical behavior of component i. Set $\underline{X}(t) = (X_1(t),\ldots, X_n(t))$. We define

$$T_{ij} = \inf\{t \geq 0 : X_i(t) \leq j\}$$

$$T_k = \inf\{t \geq 0 : \phi(\underline{X}(t)) \leq k\}$$

for $i = 1,\ldots, n$ and $j, k = 0,1,\ldots, M-1$.

(3.14) Theorem. For $k = 0,1,\ldots, M-1$,

$$T_k = \max_{\underline{x}\in U_{k+1}} \; \min_{(i,j)\in U_{k+1}(\underline{x})} T_{i,j-1}$$

$$= \min_{\underline{x}\in L_k} \; \max_{(i,j)\in L_k(\underline{x})} T_{ij}$$

Proof. First we observe that $\phi(\underline{X}(t)) \leq k$ if and only if $\phi_{k+1}(\alpha(\underline{X}(t))) = 0$. Consequently, $T_k = \tau^{k+1}$ where $\tau^{k+1} = \inf\{t \geq 0 : \phi_{k+1}(\alpha(\underline{X}(t))) = 0\}$. But from the results in the binary case,

$$\tau^{k+1} = \max_{\underline{x}\in U_{k+1}} \; \min_{(i,j)\in U_{k+1}(\underline{x})} \tau_{ij} \,,$$

where $\tau_{ij} = \inf\{t \geq 0 : \alpha_{ij}(\underline{X}(t)) = 0\}$. Since

$$\tau_{ij} = \inf\{t \geq 0 : \alpha_{ij}(\underline{X}(t)) = 0\} = \inf\{t \geq 0 : X_i(t) < j\}$$

$$= \inf\{t \geq 0 : X_i(t) \leq j-1\} = T_{i,j-1},$$

we are done. The second half follows similarly.

§4. Some remarks about coherence assumptions.

As was remarked earlier, EPS [5] and Griffith [6] also studied the deterministic properties of multistate monotone structure functions in the finite state case. Besides the basic monotonicity assumption, however, they assumed that $\Phi(k) = k$ for all $k \in S$ plus a type of coherence assumption. In [6], Griffith delineated three distinct coherence conditions, which we list below.

(SC): Φ is said to be <u>strongly coherent</u> if for any component i and any level j, there exists $\underline{x}$ such that $\Phi(j_i : \underline{x}) = j$ while $\Phi(\ell_i : \underline{x}) \neq j$ for $\ell \neq j$.

(C): Φ is said to be <u>coherent</u> if for any component i and any level $j \geq 1$, there exists $\underline{x}$ such that $\Phi((j-1)_i : \underline{x}) < \Phi(j_i : \underline{x})$.

(WC): Φ is said to be <u>weakly coherent</u> if for any component i, there exists $\underline{x}$ such that $\Phi(0_i : \underline{x}) < \Phi(M_i : \underline{x})$.

EPS [5] assumed condition (SC) for their class whereas Griffith [6] showed that all of the results of [5] hold under the assumption of (C), but some are false under (WC). Loosely speaking, condition (SC) says that every level of every component is relevant to the same level of the system Φ; condition (C) says that every level of every component is relevant to the system Φ; condition (WC) says that every component is relevant to the system Φ.

In terms of the decomposition (3.10), we can paraphrase the above as follows. Let $R = \bigcup_{k=1}^{M} \bigcup_{\underline{x} \varepsilon U_k} U_k(\underline{x})$. Then Φ is coherent if and only if for every i and j, y_{ij} is relevant to some ϕ_k. Φ is weakly coherent if and only if for every i, there exists j such that $(i,j) \varepsilon R$; i.e., for every i, there exists j such that y_{ij} is relevant to some ϕ_k. The condition of strong coherence and the conditions $\Phi(\underline{k}) \geq k$, $\Phi(\underline{k}) \leq k$ for $k \varepsilon S$ can be similarly rephrased, but since they are somewhat more complicated to state, and lose their intuitive content, we will not state them here.

(4.1) Theorem. Let Φ be a multistate monotonic structure function on $S = \{0,1,\ldots, M\}$. Then

(i) $\Phi(\underline{x} \vee \underline{y}) = \Phi(\underline{x}) \vee \Phi(\underline{y})$ for all $\underline{x}, \underline{y} \varepsilon S^n$ if and only if $\Phi(\underline{x}) = \max_i h_i(x_i)$ where $h_i(j) = \Phi(j_i; \underline{0})$.

(ii) $\Phi(\underline{x} \wedge \underline{y}) = \Phi(\underline{x}) \wedge \Phi(\underline{y})$ for all $\underline{x}, \underline{y} \varepsilon S^n$ if and only if $\Phi(\underline{x}) = \min_i H_i(x_i)$ where $H_i(j) = \Phi(j_i; \underline{M})$.

The proofs are very simple and so are omitted. Thus in case (i), Φ must be a parallel system

$$-\left[\begin{array}{c} h_1(x_1) \\ h_2(x_2) \\ \vdots \\ h_n(x_n) \end{array}\right]-$$

while in case (ii), Φ must be a series system

$$- H_1(x_1) - H_2(x_2) - \cdots - H_n(x_n) - .$$

In EPS [5], under the assumption of (SC), it was concluded that $h_i(j) = j$ and $H_i(j) = j$ for all $i = 1, \ldots, n$, $j = 0, \ldots, M$. Griffith [6] concluded the same result assuming the weaker condition (C). Griffith also showed that the result was false under (WC).

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Two types of multidimensional processes are defined. The first of these generalizes a univariate IFRA process due to Ross and relates to a multivariate concept of IFRA due to Esary and Marshall. The second of these relates to a multivariate concept of IFRA due to the present authors. Decompositions for multistate monotone structure functions are given and behavior of nonincreasing stochastic processes such as those given above is analyzed. Various coherence assumptions for multistate systems are also analyzed.

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